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On Rado numbers for $\sum_{i=1}^{m-1} a_i x_i = x_m$

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Abstract

For all integers $m \geqslant 3$ and all natural numbers $a_1, a_2, \dots, a_{m-1}$, let $n = R(a_1, a_2, \dots, a_{m-1})$ represent the least integer such that for every 2-coloring of the set $\{1, 2, \dots, n\}$ there exists a monochromatic solution to

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m.$$

Let $t = \min\{a_1, a_2, \dots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$. In this paper it is shown that whenever $t = 2$,

$$R(a_1, a_2, \dots, a_{m-1}) = 2b^2 + 9b + 8.$$

It is also shown that for all values of t,

$$R(a_1, a_2, \dots, a_{m-1}) \geqslant tb^2 + (2t^2 + 1)b + t^3.$$

Keywords: Rado numbers; Schur numbers; Ramsey theory; Colorings

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1. Introduction

Let $[a, b]$ represent the set $\{x \in \mathbb{N} \mid a \leqslant x \leqslant b\}$. A function $\Delta : [1, n] \to [0, k-1]$ is called a *k-coloring* of the set $[1, n]$. If L is a system of equations in m variables, then we say that a solution $\{x_1, x_2, \dots, x_m\}$ to L is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, Issai Schur [22] proved that for every integer $k \geqslant 2$, there exists a least integer $n = S(k)$ such that for every k-coloring of the set $[1, n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_3.$$

The integers $S(k)$ are called *Schur numbers*. It is known that $S(2) = 5$, $S(3) = 14$, and $S(4) = 45$, but the Schur numbers for $k \geqslant 5$ are unknown [23]. Richard Rado, who was a student of Schur, generalized the work of Schur to arbitrary systems of linear equations. Rado was able to find necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors [6,15–17]. For a given system of linear equations L, the least integer n, provided that it exists, such that for every coloring of the set $[1, n]$ with k colors, there exists a monochromatic solution to L, is referred to as the *k-color Rado number* (or *k-color generalized Schur number*) for the system L. If such an integer n does not exist, then the k-color Rado number for the system L is infinite. In this paper, we confine our attention to single equations.

The results of Rado may tell us that a given Rado number is finite, but they will not tell us what this number is. In recent years there has been considerable interest in finding the exact Rado numbers for particular linear equations and in several other closely related problems [2–5,7–10,12–14,18–21]. In 1982, Albrecht Beutelspacher and Walter Brestovansky [1] considered the equation

$$x_1 + x_2 + \cdots + x_{m-1} = x_m.$$

They were able to show that the 2-color Rado number for this equation is $m^2 - m - 1$ for every integer $m \geqslant 3$. In this paper we consider a generalization of this equation.

Definition. For all integers $m \geqslant 3$ and all natural numbers $a_1, a_2, \dots, a_{m-1}$, let $L(a_1, a_2, \dots, a_{m-1})$ represent the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m.$$

Let $R(a_1, a_2, \dots, a_{m-1})$ represent the 2-color Rado number for this equation.

One result of Rado is that the 2-color Rado numbers for the homogeneous linear equations $b_1x_1 + b_2x_2 + \cdots + b_mx_m = 0$ are finite if and only if there are at least three nonzero coefficients and both positive and negative coefficients. Hence, all values of $R(a_1, a_2, \ldots, a_{m-1})$ are finite.

Given an integer $m \geqslant 3$ and natural numbers $a_1, a_2, \ldots, a_{m-1}$, let $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$. In [11] it is shown that if $t = 1$,

$$R(a_1, a_2, \ldots, a_{m-1}) = b^2 + 3b + 1.$$

In this paper we find $R(a_1, a_2, \ldots, a_{m-1})$ whenever $t = 2$. A lower bound for $R(a_1, a_2, \ldots, a_{m-1})$ for all values of t is also found and it is conjectured that this lower bound is also an upper bound.

2. Main results

First we shall establish a relation between $R(t, b)$ and $R(a_1, a_2, \ldots, a_{m-1})$.

Theorem 1. *For all integers $m \geqslant 3$ and all natural numbers $a_1, a_2, \ldots, a_{m-1}$, if $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$, then*

$$R(t, b) \geqslant R(a_1, a_2, \ldots, a_{m-1}).$$

Proof. Let an integer $m \geqslant 3$ and natural numbers $a_1, a_2, \ldots, a_{m-1}$ be given and let $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$. Without loss of generality, we may assume that

$$t = a_1,$$

so

$$b = a_2 + a_3 + \cdots + a_{m-1}.$$

To show that $R(t, b) \geqslant R(a_1, a_2, \ldots, a_{m-1})$, we must show that every 2-coloring of the set $[1, R(t, b)]$ contains a monochromatic solution to $L(a_1, a_2, \ldots, a_{m-1})$. By definition, every 2-coloring of the set $[1, R(t, b)]$ contains a monochromatic solution to $L(t, b)$, i.e., a monochromatic 3-tuple (x_1, x_2, x_3) such that

$$tx_1 + bx_2 = x_3.$$

Now, if

$$y_1 = x_1 \quad \text{and} \quad y_2 = y_3 = \cdots = y_{m-1} = x_2 \quad \text{and} \quad y_m = x_3,$$

then

$$a_1 y_1 + a_2 y_2 + \cdots + a_{m-1} y_{m-1} = tx_1 + (a_2 + \cdots + a_{m-1}) x_2 = tx_1 + bx_2 = x_3 = y_m.$$

Hence, $(y_1, y_2, \ldots, y_m)$ is a monochromatic solution to $L(a_1, a_2, \ldots, a_{m-1})$ and the proof of Theorem 1 is complete. $\square$

We will now establish a lower bound for $R(a_1, a_2, \ldots, a_{m-1})$.

Theorem 2. *For all integers $m \geqslant 3$ and all natural numbers $a_1, a_2, \ldots, a_{m-1}$, if $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$, then*

$$R(a_1, a_2, \ldots, a_{m-1}) \geqslant tb^2 + \bigl(2t^2 + 1\bigr)b + t^3.$$

Proof. Let an integer $m \geqslant 3$ and natural numbers $a_1, a_2, \ldots, a_{m-1}$ be given. In order to show that

$$R(a_1, a_2, \ldots, a_{m-1}) \geqslant tb^2 + \bigl(2t^2 + 1\bigr)b + t^3,$$

we must exhibit a 2-coloring of the set $[1, tb^2 + (2t^2 + 1)b + t^3 - 1]$ that avoids a monochromatic solution to $L(a_1, a_2, \ldots, a_{m-1})$. Let $\Delta : [1, tb^2 + (2t^2 + 1)b + t^3 - 1] \to [0, 1]$ be defined by

$$\Delta(x) = \begin{cases} 0, & \text{if } 1 \leqslant x \leqslant b + t - 1, \\ 1, & \text{if } b + t \leqslant x \leqslant (b + t)^2 - 1, \\ 0, & \text{if } (b + t)^2 \leqslant x \leqslant tb^2 + (2t^2 + 1)b + t^3 - 1. \end{cases}$$

Let $(x_1, x_2, \ldots, x_m)$ be a monochromatic m-tuple. We will show that $(x_1, x_2, \ldots, x_m)$ is not a solution to $L(a_1, a_2, \ldots, a_{m-1})$. First assume that

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m) = 1.$$

Then

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} \geqslant (a_1 + a_2 + \cdots + a_{m-1})(b + t) = (b + t)^2 > x_m,$$

so $(x_1, x_2, \ldots, x_m)$ is not a solution to $L(a_1, a_2, \ldots, a_{m-1})$.

Next assume that

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m) = 0.$$

If $x_i \in [1, b + t - 1]$ for every $i \in [1, m - 1]$, then

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} \geqslant a_1 + a_2 + \cdots + a_{m-1} = b + t$$

and

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} \leqslant (a_1 + a_2 + \cdots + a_{m-1})(b + t - 1) \leqslant (b + t)^2 - 1.$$

Since $x_m \notin [b + t, (b + t)^2 - 1]$, it follows that $(x_1, x_2, \ldots, x_m)$ is not a solution to $L(a_1, a_2, \ldots, a_{m-1})$.

Now, if $x_i \in [(b+t)^2, tb^2 + (2t^2+1)b + t^3 - 1]$ for some $i \in [1, m-1]$, then

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} \geqslant b(1) + t(b+t)^2 = tb^2 + \big(2t^2+1\big)b + t^3 > x_m,$$

so $(x_1, x_2, \ldots, x_m)$ is not a solution to $L(a_1, a_2, \ldots, a_{m-1})$. Hence $\varDelta$ avoids a monochromatic solution to $L(a_1, a_2, \ldots, a_{m-1})$ and the proof is complete. □

We shall now find the Rado numbers for the linear equations $L(2, b)$.

Theorem 3. *For all integers $b \geqslant 2$,*

$$R(2, b) = 2b^2 + 9b + 8.$$

Proof. Let an integer $b \geqslant 2$ be given. From Theorem 2 it follows that

$$R(2, b) \geqslant 2b^2 + 9b + 8.$$

We will show that

$$R(2, b) \leqslant 2b^2 + 9b + 8$$

by showing that every 2-coloring of the set $[1, 2b^2 + 9b + 8]$ contains a monochromatic solution to $L(2, b)$.

Let an arbitrary 2-coloring $\varDelta : [1, 2b^2 + 9b + 8] \to [0, 1]$ be given. Without loss of generality we may assume that

$$\varDelta(1) = 0.$$

If $\varDelta(b+2) = 0$, then $(1, 1, b+2)$ is a monochromatic solution to $L(2, b)$ and we are done, so we may assume that

$$\varDelta(b+2) = 1.$$

If $\varDelta(b^2 + 4b + 4) = 1$, then $(b+2, b+2, b^2 + 4b + 4)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\varDelta\big(b^2 + 4b + 4\big) = 0.$$

If $\varDelta(2b^2 + 9b + 8) = 0$, then $(b^2 + 4b + 4, 1, 2b^2 + 9b + 8)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\varDelta\big(2b^2 + 9b + 8\big) = 1.$$

We will now consider three cases on the possible values of $\varDelta(b)$ and $\varDelta(2)$.

Case 1. Assume that $\Delta(b) = 1$.

If $\Delta(b^2 + 2b) = 1$, then $(b, b, b^2 + 2b)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta\bigl(b^2 + 2b\bigr) = 0.$$

If $\Delta(b^2 + 4b) = 1$, then $(b, b + 2, b^2 + 4b)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta\bigl(b^2 + 4b\bigr) = 0.$$

Note that since $b^2 + 3b$ is even, $(b^2 + 3b)/2$ is an integer. If $\Delta((b^2 + 3b)/2) = 0$, then $((b^2 + 3b)/2, 1, b^2 + 4b)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta\left(\frac{1}{2}\bigl(b^2 + 3b\bigr)\right) = 0.$$

Now, if $\Delta(2b^2+5) = 0$, then $(b^2+2b, 1, 2b^2+5b)$ is a monochromatic solution to $L(2, b)$, and if $\Delta(2b^2 + 5b) = 1$, then $((b^2 + 3b)/2, b + 2, 2b^2 + 5b)$ is a monochromatic solution to $L(2, b)$. Since for either possible value of $\Delta(2b^2 + 5b)$ there exists a monochromatic solution to $L(2, b)$, there exists a monochromatic solution to $L(2, b)$ whenever $\Delta(b) = 1$ and Case 1 is complete.

Case 2. Assume that $\Delta(b) = 0$ and $\Delta(2) = 1$.

If $\Delta(b^2 + 2b) = 0$, then $(b, b, b^2 + 2b)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta\bigl(b^2 + 2b\bigr) = 1.$$

If $\Delta(2b^2 + 6b) = 1$, then $(b^2 + 2b, 2, 2b^2 + 6b)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta\bigl(2b^2 + 6b\bigr) = 0.$$

Now, if $\Delta(2b + 4) = 0$, then $(b, 2b + 4, 2b^2 + 6b)$ is a monochromatic solution to $L(2, b)$, and if $\Delta(2b + 4) = 1$, then $(2, 2, 2b + 4)$ is a monochromatic solution to $L(2, b)$. Since for either possible value of $\Delta(2b + 4)$ there exists a monochromatic solution to $L(2, b)$, there exists a monochromatic solution to $L(2, b)$ whenever $\Delta(b) = 0$ and $\Delta(2) = 1$ and Case 2 is complete.

Case 3. Assume that $\Delta(b) = 0$ and $\Delta(2) = 0$.

If $\Delta(2b + 2) = 0$, then $(2b + 2, b, b^2 + 4b + 4)$ is a monochromatic solution to $L(2, b)$, so we may assume that

$$\Delta(2b + 2) = 1.$$

If $\Delta(b^2+6b+4)=1$, then $(2b+2, b+2, b^2+6b+4)$ is a monochromatic solution to $L(2,b)$, so we may assume that

$$\Delta\big(b^2+6b+4\big)=0.$$

Note that since b^2+5b+4 is even, $(b^2+5b+4)/2$ is an integer. If $(\Delta((b^2+5b+4)/2)=0$, then $((b^2+5b+4)/2, 1, b^2+6b+4)$ is a monochromatic solution to $L(2,b)$, so we may assume that

$$\Delta\left(\frac{1}{2}\big(b^2+5b+4\big)\right)=1.$$

If $\Delta(2b^2+7b+4)=1$, then $((b^2+5b+4)/2, b+2, 2b^2+7b+4)$ is a monochromatic solution to $L(2,b)$, so we may assume that

$$\Delta\big(2b^2+7b+4\big)=0.$$

If $\Delta(b+4)=0$, then $(2,1,b+4)$ is a monochromatic solution to $L(2,b)$, so we may assume that

$$\Delta(b+4)=1.$$

Now, if $\Delta(2b+7)=0$, then $(2, 2b+7, 2b^2+7b+4)$ is a monochromatic solution to $L(2,b)$, and if $\Delta(2b+7)=1$, then $(b+4, 2b+7, 2b^2+9b+8)$ is a monochromatic solution to $L(2,b)$. Since for either possible value of $\Delta(2b+7)$ there exists a monochromatic solution to $L(2,b)$, there exists a monochromatic solution to $L(2,b)$ whenever $\Delta(b)=0$ and $\Delta(2)=0$ and Case 3 is complete.

Since in each case it was shown that Δ contains a monochromatic solution to $L(2,b)$, it follows that

$$R(2,b)\leqslant 2b^2+9b+8.$$

Since a lower bound of $2b^2+9b+8$ was previously shown, it follows that

$$R(2,b)=2b^2+9b+8$$

and the proof is complete. □

We shall now find all Rado numbers for the linear equations $L(a_1,a_2,\ldots,a_{m-1})$ when the least coefficient is 2.

Theorem 4. *For all integers $m\geqslant 3$ and all natural numbers $a_1,a_2,\ldots,a_{m-1}$, if $2=\min\{a_1,a_2,\ldots,a_{m-1}\}$ and $b=a_1+a_2+\cdots+a_{m-1}-2$, then*

$$R(a_1,a_2,\ldots,a_{m-1})=2b^2+9b+8.$$

Proof. Let $m \geqslant 3$ and $a_1, a_2, \ldots, a_{m-1}$ be given. Under the hypotheses, it follows from Theorems 1 and 3 that

$$R(a_1, a_2, \ldots, a_{m-1}) \leqslant R(2, b) = 2b^2 + 9b + 8.$$

From Theorem 2 it follows that

$$R(a_1, a_2, \ldots, a_{m-1}) \geqslant 2b^2 + \left(2 \cdot 2^2 + 1\right)b + 2^2 = 2b^2 + 9b + 8. \qquad \square$$

Based on the above results and computer experiments for small values of m and t, the authors make the following conjecture.

Conjecture. *For all integers $m \geqslant 3$ and natural numbers $a_1, a_2, \ldots, a_{m-1}$, if $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \cdots + a_{m-1} - t$, then*

$$R(a_1, a_2, \ldots, a_{m-1}) = tb^2 + \left(2t^2 + 1\right)b + t^3.$$

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